Rework

This is an extra credit opportunity, available to everyone on both grading scales.

Exam 2 made it clear that many of us need more time with the material in Chapters 5–7. These chapters introduce a lot of new ideas, and ideas like these don't sink in from reading the notes once or following along in class. They take sitting down with the material and working through it yourself, slowly, until each one makes sense.

This assignment is a chance to do that. In a way, it is a second try at a study guide for the Exam 2 material, but this time I'll guide you through what should go in it. Each section below covers one kind of object from these chapters, and asks you to explain it in general and then work through a lot of examples.

It is worth 6 points on Exam 2 (a full letter grade), as pure extra credit. It is also a lot of work: a serious paper working through these chapters, and then a 10-minute conversation with me about what you wrote.

If you scored 50 or more out of 60 and feel that you have a good grasp of this material, but would still like extra credit, talk to me or send me a message, and we can come up with an alternative together.

I have two goals for you here. The first is to really learn this material. The concepts in these chapters (spaces whose points are functions, maps between those spaces, and the matrices that represent those maps) are the foundation for abstract mathematics, and the rest of the course continues to speak this language.

The second is for you to see what studying new material can look like. Being able to compute with a new idea is only the start. You also want to know what kind of thing it is, how it connects to the ideas around it, and what it looks like in many different examples. Working through material this way is how you make room in your mind for new concepts, and it works in any course.

Write your paper in full paragraphs with complete sentences. In each section, start with the general explanation, then give each example asked for and explain it. In every example, say what kind of object each thing is: a number, a vector, a function, a vector-valued function, a set, a map, a matrix, or a matrix-valued function. This will feel repetitive, and that is on purpose. Keeping track of what kind of object you are working with is exactly what this assignment is meant to build.

Vector Spaces, Linear Maps, and Affine Spaces

The ideas:

- What is a vector space, and what is a subspace? Explain how to check whether a set of vectors, or a set of functions, is a subspace, and what it takes to show that a set is not one.

- What do linear independence, a basis, and dimension mean? Explain how to check whether a list of vectors, or a list of functions, is linearly independent.

- What is a linear map? What are its kernel and its image, and which space does each one live in? State the rank–nullity theorem, and explain what it says.

- How do you write the matrix of a linear map in chosen bases? Explain why its columns are the coordinates of the images of the basis vectors. Why does composing two maps correspond to multiplying their matrices? Why does an isomorphism have an invertible matrix?

- What is an affine space? Explain why the solutions of a linear equation $T(v) = w$ form one: a particular solution plus the kernel.

Examples:

- A subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ that is a subspace, and one that is not. Then a set of functions that is a subspace of a space of functions, and one that is not.

- A list of vectors that is linearly independent, and one that is not. Then the same for lists of functions.

- Write down the standard basis for the space P_2 of polynomials of degree at most two. Now write down *another* basis for the same space. Express one specific (nontrivial!) polynomial in both bases.

- A linear map whose inputs are polynomials. Find its kernel and its image, and check that rank–nullity holds.

- The matrix of that map in bases you choose. Use the matrix to find the output for one specific input, and check your answer by applying the map directly.

- Two linear maps that can be composed. Find the matrix of the composition directly, and then by multiplying their matrices.

- A linear isomorphism, its matrix, and the inverse of that matrix. Explain what the inverse matrix does.

- An equation $T(v) = w$ for a linear map T. Find one particular solution and the kernel, and describe every solution as an affine space.

Solutions and Solution Sets

The ideas:

- What is the solution set of a differential equation, and what is its ambient space? Compare and contrast a single equation with a system. In each case, what kind of object is one solution, and what space do the solutions live in? How are the two cases alike, and how are they different?
- We know the solution sets of $y' = y$ and of the spring $x'' = -x$ exactly. Explain why the first is a line and the second is a plane in a space of functions.
- The logistic equation $y' = y(1 - y)$ also has one solution for each initial value $y(0)$, so its solutions form a one-parameter family. Why isn't that family a line?
- Why are spaces of functions such as $C^k(I)$ vector spaces? Why are they infinite-dimensional?

Examples:

- A first-order equation whose solution set is a line. Find the solution set and explain why it is a line.
- A second-order equation whose solution set is a plane. Find the solution set and explain why it is a plane.
- A system of two first-order equations. Describe its solution set and its ambient space.
- A nonlinear equation. Find two of its solutions, and show that their sum, or a multiple of one of them, is not a solution.
- A list of functions showing that $C^k(I)$ is infinite-dimensional. Explain why your list shows this.

First-Order Form and Initial Conditions

The ideas:

- Explain how to write any equation in first-order form: how to choose the state, how to write the system, and what an initial condition prescribes.
- An equation and its first-order form are *different* equations. A solution of a second-order equation is a single function y, while a solution of its first-order form is a vector-valued function (y, v). Explain how to go back and forth: how a solution of the original equation gives a solution of the system, and how a solution of the system gives back a solution of the original equation. Why do the two solution sets carry exactly the same information? For a linear equation, why is this correspondence an isomorphism of vector spaces?
- What is the initial-condition space $\mathcal{I}$, and how do you find it for a given equation?

Examples:

- A second-order equation. Write it in first-order form, name the state, and find $\mathcal{I}$. Give one solution of the original equation, find the corresponding solution of the system, and then convert it back.
- A system of two first-order equations that comes from a second-order equation. Recover the second-order equation, and convert a solution in both directions.
- A system of two first-order equations that does not come directly from a second-order equation. Say what its state is, and find $\mathcal{I}$.
- A third-order equation. Write it in first-order form, name the state, and find $\mathcal{I}$.
- A system in which the unknowns appear to different orders. Write it in first-order form, say what an initial condition prescribes, and find $\mathcal{I}$.

The Evaluation Map and the Solution Map

The ideas:

- State the existence and uniqueness theorem for systems. What are its hypotheses, and what does it let you conclude?
- What is the evaluation map? What does it take in, and what does it give back? Why does uniqueness make it one-to-one? Why can it fail to be onto?
- What is the solution map, and on which initial conditions is it defined?
- Explain how to use existence and uniqueness to prove that you have found *all* the solutions of a differential equation, once you have found one for every possible initial condition.

Examples:

- An equation in first-order form. Check the hypotheses of the existence and uniqueness theorem.
- Apply the evaluation map to several specific solutions of a second-order equation, and of a system.
- An equation whose evaluation map is not onto on a given interval. Say which initial conditions have no solution in $\mathcal{S}$, and where the solution map is defined.
- An equation where uniqueness fails. Give two different solutions with the same initial condition, and explain why evaluation is not one-to-one.
- A family of solutions containing one solution for each initial condition. Use uniqueness to prove that it contains every solution.

Linear Equations

The ideas:

- What is a linear operator? How do you check whether a given operator is linear?
- What is a linear differential equation $L[y] = g$? What do homogeneous and forced mean? How do you decide whether a given equation is linear?
- How do you write a linear first-order system in matrix form, $X' = A(t)X + g(t)$, and read off $A(t)$ and $g(t)$?
- Why do the solutions of a homogeneous linear equation form a vector space? Why do the solutions of a forced linear equation form an affine space $u_p + \ker L$?
- State the existence and uniqueness theorem for linear systems. What does it promise that the general theorem does not?
- Why is evaluation a linear isomorphism for a homogeneous linear equation, so that $\dim \mathcal{S} = \dim \mathcal{I}$? How does this let you find the dimension of the solution space without solving the equation?

Examples:

- An operator on a space of functions that is linear, and one that is not. Check each one.
- Several equations, both scalar equations and systems, some linear and some not. For each linear one, identify L and g, and say whether it is homogeneous or forced.
- A linear system with coefficients that depend on time. Write it in matrix form and identify $A(t)$ and $g(t)$.
- A homogeneous linear equation. Check directly that sums and multiples of its solutions are solutions.
- A forced linear equation. Find one particular solution, and describe every solution as an affine space. (It's ok if the example is easy, or one from the book! This is *studying*, after all!)
- A nonlinear equation whose solutions are only promised near t_0, and a linear equation whose solutions are promised on a whole interval. Explain the difference.
- Several homogeneous linear equations and systems. Find the dimension of each solution space without solving.

Bases of Solutions and the Matrix of Evaluation

The ideas:

- How do you find every solution of a homogeneous linear equation by finding dim $\mathcal{I}$ independent solutions? How do you find every solution of a forced one?
- How do you check whether solutions form a basis by checking whether their initial conditions do?
- Given a basis of solutions, how do you write the matrix M of the evaluation map? How do you use M^{-1} to solve initial value problems?
- What are the standard solutions, and why is their evaluation matrix the identity?

Examples:

- A homogeneous second-order equation. Find two solutions, check that they form a basis using their initial conditions, write M, find M^{-1}, and use it to solve an initial value problem. Check your answer.
- A homogeneous linear system. Choose a basis of solutions whose evaluation matrix is diagonal, and another basis whose evaluation matrix is not. Can you choose one whose evaluation matrix is the identity?
- A homogeneous third-order equation, such as $y''' = y'$, or $x''' = x''$ from Problem Set 5 (or try others!). Find three solutions and explain why they form a basis.
- A forced linear equation. Find every solution, and solve an initial value problem.

The Flow Matrix

The ideas:

- What is the flow matrix of a homogeneous linear system, and how is it built from the standard solutions? Why is $X(t) = \Phi(t)X_0$?
- Why does this require the *standard* solutions, when we write initial conditions in the usual basis $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, and so on?
- Fix a time t. The matrix $\Phi(t)$ takes in an initial condition and gives back the state at time t, so its domain and its range are both the space of states. Explain how this lets us think of solving the system as evolving every state at once. Then explain how an individual solution is what you get by picking one initial state and following it over time.

Examples:

- The flow matrix of a second-order equation, written in first-order form. Explain what its columns are.

- The flow matrix of a system of two first-order equations.
- For one of these, use the flow matrix to solve an initial value problem. Then find $\Phi(t)$ at one specific time, and explain what that matrix does to initial conditions.
- A system whose solution from a general initial condition (x_0, y_0) you can write down. Explain why that formula already is $\Phi(t)X_0$, and read off the columns of $\Phi(t)$.
- The flow matrix of the spring. Describe what $\Phi(t)$ does to the whole plane of states, and what happens after time 2π.
- Running a solution backward. The flow matrix carries initial conditions forward in time, but it can also take us back. For one of your systems, pick a state and a later time t_1, and find the initial condition whose solution arrives at that state at time t_1. Check your answer by following that solution forward to t_1. Explain why this always works: why is there exactly one initial condition whose solution reaches a given state at time t_1?

The Matrix Exponential

The ideas:

- Why does the flow matrix of a linear system satisfy the matrix equation $\Phi' = A(t)\Phi, \Phi(t_0) = I$?
- How do you find the power series of the exponential from the differential equation $y' = ay$, $y(0) = 1$, by matching coefficients? Why does this series define e^x for every real x, and not just for rational x?
- What is the definition of the matrix exponential e^{tA}, and how does the same matching of coefficients produce it?
- Why is e^{tA} the flow matrix of $X' = AX$, so that the solution from $X(t_0) = X_0$ is $e^{(t-t_0)A}X_0$?
- How does uniqueness prove the law of exponents $e^{(t+s)A} = e^{tA}e^{sA}$? Why does it give $(e^{tA})^{-1} = e^{-tA}$?
- How do you compute e^{tA} when the powers of A follow a pattern?
- Why can $e^{t(B+C)}$ differ from $e^{tB}e^{tC}$ when $BC \neq CB$?

Examples:

- Use the law of exponents for the function defined by $E' = E, E(0) = 1$ to find $E(-1)$ and $E(1/2)$ in terms of $e = E(1)$. Explain how this shows that the series agrees with the powers and roots you already knew.

- e^{tA} for a multiple of I, for a diagonal matrix, for a matrix whose powers eventually vanish, and for a matrix whose powers cycle and produce cosine and sine. For each one, check that it solves $\Phi' = A\Phi$, $\Phi(0) = I$, and use it to solve an initial value problem.
- For one of your matrices, check the law of exponents and the formula for the inverse directly, by multiplying matrices.
- Two matrices B and C with $BC \neq CB$. Show that e^{B+C} and $e^B e^C$ are different.

Turning It In

When you finish, arrange a time to come turn it in to me, along with a 10-minute conversation about what you've learned. This can be during office hours, or at a different time that we both arrange. In the conversation, I'll ask you to explain parts of what you wrote.

This is not a second exam. It is a chance to talk through the material together. Explaining an idea out loud to someone else is one of the best ways to find out whether you really understand it, and the conversation gives me a chance to help with anything that is still unclear. It is also why the paper needs to be your own work: the conversation is about what you wrote, and if you worked through the ideas yourself, you will be ready for it.