Assignment 8

Due Friday, November 20

1.

Recall the recipe from Chapter 10 for building K when equal masses m are joined by springs. Start from the zero matrix. A spring of stiffness k joining masses i and j adds k/m to the diagonal entries K_{ii} and K_{jj}, and adds $-k/m$ to both K_{ij} and K_{ji}. A spring joining mass i to a fixed support adds k/m to K_{ii}.

Two equal masses, each with $m = 1$, are attached to fixed supports by springs of stiffness 1. They are also joined to each other by a weak spring of stiffness κ, where $\kappa > 0$ is small. Their displacements q_1 and q_2 satisfy $q'' = -Kq$.

(a) Use the recipe to write down K. Find its eigenvalues and eigenvectors, and the angular frequencies of the two modes. Explain physically why the frequency of the mode in which the masses move together does not depend on κ.

(b) Mass 1 is pulled to displacement 1 while mass 2 is held at 0, and both are released from rest. Write the initial displacement as a combination of the two mode shapes, and find $q(t)$.

(c) Use the identities

$$\cos A + \cos B = 2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2}, \qquad \cos A - \cos B = -2\sin\tfrac{A+B}{2}\sin\tfrac{A-B}{2}$$

to write each displacement as a product of two oscillating factors. When κ is small, which factor oscillates quickly and which slowly? Find the first time $T > 0$ at which the slow factor of q_1 is zero. What is mass 2 doing around that time?

(d) Take $\kappa = 0.1$. Find T, and about how many swings mass 1 makes before it hands its motion to mass 2. Then use the approximation $\sqrt{1+2\kappa} \approx 1 + \kappa$ for small κ to estimate T in terms of κ. What happens to the transfer as the coupling gets weaker? What happens when $\kappa = 0$?

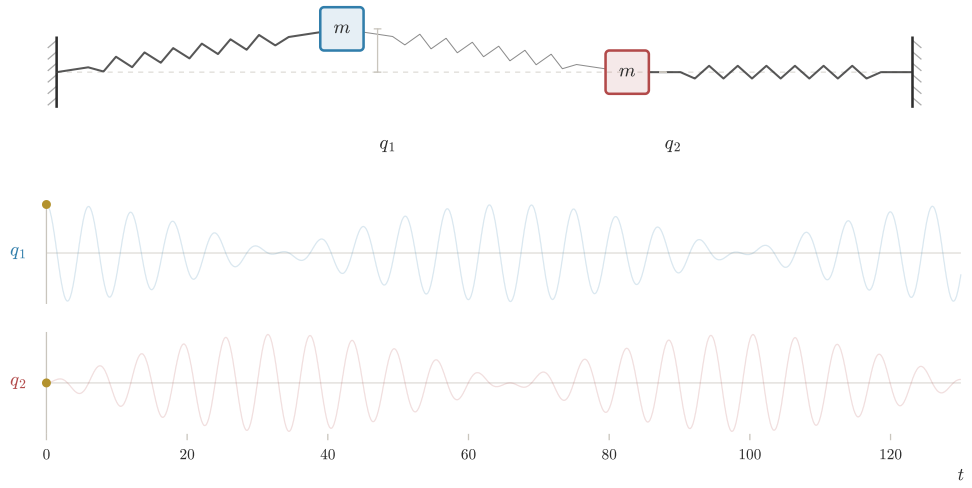


Figure 1 Mass 1 starts at displacement 1, mass 2 starts at 0, and both start from rest. The thin middle spring is the weak one. Below, the traces record q_1 and q_2 over time. Press Play to watch, and move the slider to change κ.

2.

Three equal masses, each with $m = 1$, are attached to fixed supports by springs of stiffness 1. Each pair of masses is also joined by a spring of stiffness 1, so every mass is connected to both of the others. Their displacements q_1, q_2, and q_3 satisfy $q'' = -Kq$.

(a) Use the recipe recalled in Problem 1 to write down K. Show that $(1, 1, 1)$ is an eigenvector of K, and find the angular frequency of this mode. Explain physically why the springs joining the masses do not affect this frequency.

(b) Show that every vector $v = (a, b, c)$ with $a + b + c = 0$ is an eigenvector of K, all with the same eigenvalue. Find two such mode shapes that are orthogonal to each other. What are the angular frequencies of the system?

(c) The masses are at rest at equilibrium when mass 1 is struck, so that

$$q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad q'(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find $q(t)$.

(d) Show that masses 2 and 3 move identically. Explain why we should expect this from the arrangement of the springs. Does this motion ever return exactly to its starting state? If so, after how long?

3.

This problem compares the equilibria and periodic orbits of linear and nonlinear equations.

(a) Let A be a 2×2 matrix, and suppose X_0 is an equilibrium of $X' = AX$. Show that every multiple cX_0 is also an equilibrium. Conclude that a linear system has either exactly one equilibrium or infinitely many. Why does it always have at least one?

(b) In one variable, a linear equation $y' = ay$ has either exactly one equilibrium or every number as an equilibrium. A nonlinear equation $y' = f(y)$ has much more freedom. Find a function f for which the equilibria of $y' = f(y)$ are:

1. exactly five numbers;
2. exactly the integers;
3. *(Optional challenge)* exactly the numbers $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ in the interval $[0, 1]$.

Extra challenge: can you find such an equation that also satisfies the hypotheses of the existence and uniqueness theorem?

(c) Let A be a 2×2 matrix, and let $X(t)$ be a nonconstant periodic solution of $X' = AX$ with period T, so that $X(t + T) = X(t)$ for all t. Show that the system has infinitely many periodic orbits. *(Hint: consider multiples $cX(t)$. Show that they are solutions and that they are periodic.)*

4.

Two species compete for the same resources. Their populations x and y satisfy

$$x' = x(3 - 2x - y), \qquad y' = y(2 - x - y).$$

(a) The nullclines are the curves where $x' = 0$ or where $y' = 0$. Find them, and use them to find all four equilibria.

(b) Compute the Jacobian matrix $J(x, y)$. Evaluate it at the equilibrium where both species are present, and classify that equilibrium. Do populations that start near it move toward it or away from it?

5.

In the system of Problem 4, suppose only species x is present, at the equilibrium $(3/2, 0)$, when a few individuals of species y arrive. Write the populations as $x = 3/2 + u$ and $y = v$, where u and v are small.

(a) Use the Jacobian at $(3/2, 0)$ to write the linear system that approximately describes u and v.

(b) The equation for v does not involve u. Solve it. Does the population of species y grow or die out?

6.

A pendulum swinging without friction satisfies $\theta'' = -\sin\theta$, where θ is the angle from straight down.

(a) Write this as a first-order system for θ and $\omega = \theta'$. Find all of its equilibria, and say what they mean for the pendulum.

(b) Compute the Jacobian at $(0, 0)$ and at $(\pi, 0)$. Show that near $(0, 0)$ the linear system is the spring $u'' = -u$, where u is the small displacement of θ from the equilibrium. What second-order equation do you get near $(\pi, 0)$? Classify both linear flows.