

Assignment 6

Due Wednesday, October 21

1.

Consider the system

$$x' = x + 2y, \quad y' = 3x + 2y.$$

- (a) Decide which case this system falls into: two independent real eigenvectors, one, or none.
- (b) Find the solution with initial condition $x(0) = x_0, y(0) = y_0$. Then find the solution with $x(0) = 1, y(0) = 4$.

2.

Consider the system

$$x' = x - 4y, \quad y' = 2x - 3y.$$

- (a) Decide which case this system falls into: two independent real eigenvectors, one, or none.
- (b) Find the solution with initial condition $x(0) = x_0, y(0) = y_0$. Then find the solution with $x(0) = 1, y(0) = 1$.

3.

Consider the system

$$x' = -x - 4y, \quad y' = x - 5y.$$

- (a) Decide which case this system falls into: two independent real eigenvectors, one, or none.

(b) Find the solution with initial condition $x(0) = x_0, y(0) = y_0$. Then find the solution with $x(0) = 3, y(0) = 1$.

4.

The eigenvalues tell us how a flow scales its eigenvector directions. In this problem, we investigate what they tell us about **area**.

(a) Return to the system in Problem 1. Consider the parallelogram whose sides are the eigenvectors

$$v_1 = (2, 3), \quad v_2 = (1, -1).$$

Under the flow, by what factor does each side scale after time t ? By what factor does the area change? Explain why the area grows even though one side contracts.

(b) A linear map with matrix M multiplies areas by $|\det M|$. For the system $X' = AX$, the flow matrix is e^{tA} , so its determinant tells us how the flow changes area.

Suppose A has two independent real eigenvectors. Show that

$$\det(e^{tA}) = e^{t \operatorname{tr} A}.$$

(Hint: use diagonalization and the identity $\det(XY) = \det(X) \det(Y)$ to simplify your calculation.)

(c) Give an example of a system $X' = AX$ whose flow preserves area but has trajectories that travel arbitrarily far from the origin. You may choose A to be diagonal. Explain how your example has both properties.

The formula in part (b) also holds when A has only one independent real eigenvector or none. For every real 2×2 matrix, the trace determines the area factor: positive trace gives exponential area growth, negative trace gives exponential area decay, and zero trace gives area preservation.

5.

Consider the family of systems

$$x' = -x + cy, \quad y' = -2y,$$

where $c > 0$ is a constant.

(a) Find the eigenvalues and classify the flow. Explain why every solution approaches the origin as $t \rightarrow \infty$, no matter how large c is.

(b) Now follow the solution starting at

$$x(0) = 0, \quad y(0) = 1.$$

You may use the eigenvectors $(1, 0)$ and $(c, -1)$. Write the initial state as a combination of these vectors, then find $x(t)$ and $y(t)$.

When does $x(t)$ reach its largest value for $t \geq 0$? How does that value depend on c ? Use your answer to choose a system whose trajectory starts one unit from the origin, travels more than 100 units away, and eventually returns to the origin.

Every system in this family is a sink, yet you have shown that a state can travel far from the origin before returning. Being a sink describes what happens in the long run; it does not require trajectories to move steadily closer to the origin.