

Assignment 5

Due Thursday, October 8

1. Solving from a basis we find

Consider the equation

$$y'' - 3y' + 2y = 0.$$

(a) Before looking for any solutions, explain why the solution space $\mathcal{S}$ of this equation is two-dimensional. Then look for solutions by guessing and checking: try an exponential $y = e^{rt}$. Find two linearly independent solutions.

(b) Use your two solutions from the previous part as an ordered basis $\mathcal{B}$ of the solution space. Write down the evaluation map

$$E_0 : \mathcal{S} \longrightarrow \mathbb{R}^2$$

onto the space of initial conditions. Using the basis $\mathcal{B}$ for $\mathcal{S}$ and the standard basis $(1, 0), (0, 1)$ for $\mathbb{R}^2$, express E_0 as a matrix.

(c) The inverse of the evaluation map is the *solution map* $E_0^{-1} : \mathbb{R}^2 \rightarrow \mathcal{S}$. It takes an initial condition (y_0, v_0) to the solution starting from it. In these same bases, write down the matrix of the solution map, and use it to write the solution from (y_0, v_0) .

(d) Use your work to solve $y'' - 3y' + 2y = 0$ with each of the following initial conditions:

- $y(0) = 3, y'(0) = 4;$
- $y(0) = 1, y'(0) = -1;$
- $y(0) = 0, y'(0) = 2.$

2. Discovering the Hyperbolic Functions

Consider the equation

$$y'' = y.$$

In class, we found the basis $\mathcal{B} = (e^t, e^{-t})$ for its solution space $\mathcal{S}$, and the evaluation map taking each solution to its initial data:

$$E_0 : ae^t + be^{-t} \mapsto (a + b, a - b).$$

Form its matrix M , using the basis $\mathcal{B}$ for $\mathcal{S}$ and the standard basis for $\mathbb{R}^2$. To solve an initial-value problem in this basis, we have to invert M . In this problem we look for a better basis.

(a) Find a basis (R_1, R_2) of $\mathcal{S}$ in which the matrix of E_0 is the identity. What must $E_0(R_1)$ and $E_0(R_2)$ be? Write R_1 and R_2 as combinations of e^t and e^{-t} .

(b) Check that in the basis (R_1, R_2) , the matrix of evaluation is the identity. Explain why the matrix of the solution map is then the identity too, and write down the solution with initial condition (y_0, v_0) . Use it to solve $y'' = y$ with $y(0) = 2, y'(0) = -1$.

(c) The functions R_1 and R_2 are called the **hyperbolic cosine and sine**, written $\cosh t$ and $\sinh t$. They get this name because they parametrize a hyperbola. Just as the point $(\cos t, \sin t)$ moves along the circle $x^2 + y^2 = 1$, the point $(\cosh t, \sinh t)$ moves along the hyperbola $x^2 - y^2 = 1$.

To see this, we need to show that the hyperbolic functions satisfy the identity

$$\cosh^2 t - \sinh^2 t = 1.$$

To show that a function is constant, we can show that its derivative is zero. Then we can find which constant it is by evaluating it at a single point. Carry out this strategy for $\cosh^2 t - \sinh^2 t$:

- Show that $\cosh' = \sinh$ and $\sinh' = \cosh$.
- Use these to show that the derivative of $\cosh^2 t - \sinh^2 t$ is zero.
- Evaluate $\cosh^2 t - \sinh^2 t$ at $t = 0$ to find the constant it equals.

3. A missing solution

Consider the third-order equation

$$x''' - x'' = 0.$$

(a) Before looking for any solutions, explain why the solution space $\mathcal{S}$ is three-dimensional. Then try $x = e^{rt}$ and find every value of r that works. How many linearly independent solutions does this give? Why does the dimension of $\mathcal{S}$ tell you that these cannot be all of the solutions?

(b) Look for a third solution by trying powers $x = t^n$. Which powers work?

Then check that your three solutions are linearly independent. You do not need to work with the functions directly. The evaluation map $E_0(x) = (x(0), x'(0), x''(0))$ is a linear isomorphism, and a linear isomorphism preserves linear independence in both directions. So solutions are linearly independent exactly when their initial data are linearly independent vectors in $\mathbb{R}^3$. Compute the initial data of your three solutions, and show that these vectors are independent.

(c) Use your three solutions as the basis $\mathcal{B}$ of $\mathcal{S}$, and the standard basis for $\mathbb{R}^3$. Express the evaluation map E_0 as a matrix. Then invert it to find the matrix of the solution map, which takes an initial condition (x_0, v_0, w_0) to its solution.

(d) Use your work to solve $x''' - x'' = 0$ with each of the following initial conditions:

- $x(0) = 1, x'(0) = 0, x''(0) = 2$;
- $x(0) = 2, x'(0) = 1, x''(0) = 1$;
- $x(0) = 0, x'(0) = 3, x''(0) = -1$.

4. The Hyperbolic Functions from a Matrix Exponential

We return to the equation

$$y'' = y,$$

this time solving it with the matrix exponential. Along the way, we will find the power series of the hyperbolic functions.

(a) Set $v = y'$ and write the equation as a first-order system $X' = AX$ with state $X = (y, v)$. Compute A^2, A^3 , and A^4 . What pattern do the powers of A follow?

(b) Substitute your pattern into the series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Collect the even and odd powers to write

$$e^{tA} = c(t)I + s(t)A,$$

where $c(t)$ and $s(t)$ are scalar power series. Find these two series, and write e^{tA} as a 2×2 matrix with entries $c(t)$ and $s(t)$.

(c) The matrix e^{tA} is the flow matrix of the system, so its first column is the state $(y(t), v(t))$ of the solution starting from $(1, 0)$. Use your work in *Discovering the Hyperbolic Functions* to identify that solution and its state. What functions are $c(t)$ and $s(t)$? What power series have you found?

5. The Exponential of a Sum

For numbers, $e^{a+b} = e^a e^b$. In this problem we test whether the same law holds for matrices, using

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(a) Compute B^2 and C^2 . Use your answers to find e^{tB} and e^{tC} from the exponential series.

(b) Compute $B + C$. Where have you seen this matrix before, and what is $e^{t(B+C)}$? Now compute the products $e^{tB} e^{tC}$ and $e^{tC} e^{tB}$. Does $e^{t(B+C)} = e^{tB} e^{tC}$? Does the order of the product matter?

(c) Why does the law fail? Let B and C now be any two square matrices of the same size. Write out the terms of $e^{t(B+C)}$ and of $e^{tB} e^{tC}$ up through t^2 . Show that the t^2 terms agree exactly when $BC = CB$. Compute BC and CB for our matrices. Why does this issue never arise for numbers?

6. A Series That Stops

Consider the third-order equation

$$x''' = 0.$$

(a) Solve this equation with initial data $x(0) = x_0$, $x'(0) = v_0$, $x''(0) = w_0$ by integrating three times.

(b) Set $v = x'$ and $w = x''$, and write the equation as a first-order system $X' = NX$ with state $X = (x, v, w)$. Compute N^2 and N^3 .

(c) Compute e^{tN} from the exponential series. Use it to solve the system from the initial state $X_0 = (x_0, v_0, w_0)$, and check that you recover your answer from the first part.