

Assignment 4

Due Wednesday, September 30

1. Subspaces and affine sets of functions

Work in the vector space $C^1(\mathbb{R})$. For each subset below, decide whether it is a vector subspace, an affine set that is not a vector subspace, or neither. Justify your answers using the definitions; when closure fails, give explicit functions showing the failure. For an affine set that is not a subspace, write it as $f_p + V$ and identify both the particular function f_p and the vector subspace V .

- (a) $\{f \in C^1(\mathbb{R}) : f(0) = 0\}$.
- (b) $\{f \in C^1(\mathbb{R}) : f(0) = 1\}$.
- (c) $\{f \in C^1(\mathbb{R}) : f(0)f(1) = 0\}$.

2. Linear maps on polynomials

Let P_2 be the space of real polynomials of degree at most two, with ordered basis $(1, t, t^2)$. Consider the map

$$T : P_2 \longrightarrow \mathbb{R}^2, \quad T(p) = (p(0), p'(0)).$$

- (a) Show that T is linear. Find its matrix using the given basis of P_2 and the standard basis of $\mathbb{R}^2$.
- (b) Find the kernel and image of T , giving a basis and dimension for each. Is T an isomorphism? Explain what information about a polynomial is lost when we record only its value and first derivative at zero.
- (c) Now record one more derivative:

$$J : P_2 \longrightarrow \mathbb{R}^3, \quad J(p) = (p(0), p'(0), p''(0)).$$

Find the matrix of J in the same polynomial basis and the standard basis of $\mathbb{R}^3$. Is this map an isomorphism? Justify your answer, and explain how to recover the polynomial from these three numbers.

3. Linear, homogeneous, or neither?

For each equation below, decide whether it is linear. If it is linear, say whether it is homogeneous or forced, and identify its coefficient functions. Write each linear equation as $L[u] = g$, giving the domain and codomain of the differential operator L . Use a suitable space $C^k(I)$, where I is an open interval. If it is nonlinear, point to a term which makes it so.

(a) $y'' + t^2 y' - (\sin t) y = 0$.

(b) $y' + 3y = e^{-t}$.

(c) $y'' + \sin y = 0$.

4. Solution sets inside function spaces

Now consider the whole set of solutions, with no initial condition imposed. In each part, work with functions defined on all of $\mathbb{R}$.

(a) For $u'' + 4u = 0$, define

$$L : C^2(\mathbb{R}) \longrightarrow C(\mathbb{R}), \quad L[u] = u'' + 4u.$$

Verify that L is linear. Explain why the solution set is $\ker L$, and prove that it is a vector subspace of $C^2(\mathbb{R})$. What is the zero vector in this space? Your argument should not require a formula for the general solution.

(b) For $u' + u = 1$, define $L[u] = u' + u$ on $C^1(\mathbb{R})$ and check that L is linear. If u and v are solutions, compute $L[u + v]$ and $L[u - v]$. Explain why the solution set is not a vector subspace, even though the equation is linear.

Find a particular solution u_p and show that the full solution set is $u_p + \ker L$: prove both that every solution has this form and that every function of this form is a solution. Describe this affine set by identifying its translation and its space of directions.

(c) For $u' = u \left(1 - \frac{u}{10}\right)$, compare $(2u)'$ with the right-hand side evaluated at $2u$. Show that the only solution u for which $2u$ is also a solution is $u = 0$. Use a nonzero equilibrium to conclude that the solution set is not a vector subspace of $C^1(\mathbb{R})$. Since it contains the zero function, explain why it cannot be an affine translate of a vector subspace either.

5. A basis of spring solutions

Let A and B be the solutions of

$$x'' + 4x = 0$$

with initial data $(A(0), A'(0)) = (1, 0)$ and $(B(0), B'(0)) = (0, 1)$.

(a) Find formulas for A and B , and verify that each satisfies the equation and its initial data.

(b) Let $\mathcal{S}$ be the solution space of this equation on $\mathbb{R}$. Consider the initial-data map

$$E_0 : \mathcal{S} \longrightarrow \mathbb{R}^2, \quad E_0(x) = (x(0), x'(0)).$$

Rewrite the equation as a first-order system and check the hypotheses of the global linear existence-and-uniqueness theorem. What does the theorem guarantee for each initial position and velocity?

(c) Show that E_0 is linear: for solutions $u, v \in \mathcal{S}$ and real numbers a, b , verify that

$$E_0(au + bv) = aE_0(u) + bE_0(v).$$

(d) Explain how existence and uniqueness give the other two properties we need:

- **Surjectivity:** Why is every vector in $\mathbb{R}^2$ the initial state of some solution in $\mathcal{S}$?
- **Injectivity:** Why must two solutions with the same initial state be the same function?

Conclude that E_0 is a linear isomorphism.

(e) Compute $E_0(A)$ and $E_0(B)$. Do these vectors form a basis of $\mathbb{R}^2$? Use the isomorphism E_0 to explain why A, B form a basis of the solution space $\mathcal{S}$.

(f) Without solving another differential equation, write down the solution $x(t)$ with $x(0) = -3$ and $x'(0) = 5$. Explain how expressing the initial state in a basis of $\mathbb{R}^2$ determines the whole function in the basis A, B .

(g) The coefficient of B in your answer is the initial velocity 5, but the coefficient of $\sin 2t$ is not. Explain.

6. From higher order to first order

Convert each second-order equation or system below into a first-order system: name the state, and write the system as $X' = F(t, X)$. When the system is linear and homogeneous, write it as $X' = AX$ with an explicit matrix.

(a) Two carts slide without friction on a track. The first is attached to a wall by a spring of stiffness k_1 , and the second is attached to the first by a spring of stiffness k_2 . If x_1 and x_2 are their displacements from rest, Hooke's law and Newton's second law give

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1), \quad m_2 x_2'' = -k_2 (x_2 - x_1).$$

Convert this pair to a first-order system and write its matrix. How many standard solutions reconstruct every motion of the carts? Which theorem from the notes guarantees that every motion exists for all time?

(b) A pendulum of length L swinging through the angle θ satisfies

$$\theta'' = -\frac{g}{L} \sin \theta.$$

Convert this to a first-order system. Is it linear? Check the hypotheses of the local existence-and-uniqueness theorem in state space, and state what it guarantees about a pendulum released from any angle with any angular velocity. Does the global theorem for linear systems apply?