

Assignment 3

Due Tuesday, September 22

1. Using a solution we know

Suppose $g(t, y)$ and its partial derivative $g_y(t, y)$ are continuous everywhere, and consider the differential equation

$$y' = (y - 2t)g(t, y) + 2.$$

- (a) Verify that $y(t) = 2t$ is a solution, regardless of the choice of g .
- (b) Check that the hypotheses of the local existence-and-uniqueness theorem hold at every point.
- (c) Let $y(t)$ be the solution with $y(0) = 1$. Suppose this solution is defined at $t = 10$. Show that $y(10) > 20$.

2. Waiting before leaving

Consider the initial-value problem

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

- (a) Verify that $y(t) = 0$ solves this initial-value problem.
- (b) For any $a \geq 0$, define

$$y_a(t) = \begin{cases} 0, & t \leq a, \\ (t - a)^2, & t > a. \end{cases}$$

Show that y_a also solves the initial-value problem. Be sure to check differentiability and the differential equation at $t = a$.

- (c) These solutions agree for a while and then separate. Why does this not contradict the existence-and-uniqueness theorem? Identify precisely which hypothesis fails.

3. How long does the solution exist?

Consider the two initial-value problems

$$y' = \frac{y^2}{e^t}, \quad y(0) = 1,$$

and

$$y' = ty^2, \quad y(0) = 1.$$

- (a) For each equation, check the hypotheses of the local existence-and-uniqueness theorem. What does it guarantee about these initial-value problems?
- (b) Solve both problems by separating variables, and find the largest interval containing 0 on which each solution exists.
- (c) Now suppose $a(t)$ is continuous for all real t , and $n \geq 2$ is an integer. Solve

$$y' = a(t)y^n, \quad y(0) = 1.$$

Your answer may involve

$$A(t) = \int_0^t a(s) ds,$$

and that is okay! Describe the largest interval containing 0 on which your solution exists, in terms of $A(t)$. Explain why the solution must remain positive throughout that interval.

4. Should we preheat the oven?

In this problem, use Newton's law of heating to compare two ways of heating an object. The measurements and oven temperature history below are illustrative.

- (a) An object initially at 20°C is placed in an oven maintained at 200°C . After 10 minutes, its temperature is 110°C . Assuming Newton's law of heating, determine the constant k . How long after entering the oven will the object reach the target temperature of 155°C ?
- (b) Now suppose an identical object is placed in the oven before it is switched on. Both initially have temperature 20°C , and the oven warms according to

$$A(t) = 200 - 180 2^{-t/5},$$

with t in minutes. Assume the same heating constant k applies and that the object does not appreciably affect the oven's temperature. Write and solve the initial-value problem for the object's temperature.

(c) How long after switching on the oven does the object reach 155°C ? Compare with part (a). Here we are comparing the time the object spends inside the oven; part (a) excludes the time spent preheating the empty oven.

5. Zombies on campus

A campus contains N people, some of whom have become zombies. Nobody enters, leaves, or dies: becoming a zombie changes a person's condition but does not change the total population. Let $Z(t)$ be the number of zombies, with time measured in days.

Suppose each zombie encounters people at a constant rate $r > 0$, and those encounters are spread randomly across the population. An encounter with a human turns that human into a zombie; an encounter with another zombie changes nothing.

(a) If there are currently Z zombies, how many humans remain? What fraction of encounters are with humans? Use this to write the rate at which one zombie creates new zombies.

(b) Use your answer to build a differential equation for $Z(t)$. Explain each factor in your equation. Check the model at $Z = 0$ and $Z = N$: what does it predict, and does that make sense? What familiar population model have you obtained?

(c) Sprinklers spray an antidote across campus. Each zombie encounters the spray once per day on average. Let $0 \leq c \leq 1$ describe its effectiveness: $c = 0.87$ means that 87% of zombies encountering the spray are cured. Cured zombies become human again.

Revise your differential equation to include the spray, and explain the new term. Does treatment change your expression for the number of humans?

(d) How effective must the spray be to make the zombie population decrease for every initial population $0 < Z(0) \leq N$? Remember that $0 \leq c \leq 1$. Is it always possible for the spray to be effective enough? If the spray is less effective than this, can your model support a constant positive population of zombies? Find that population and explain how new infections and cures balance there. You do not need to solve the differential equation.