MATH 340 · UNIVERSITY OF SAN FRANCISCO

Assignment 1

Due Thursday, September 3

Work these on your own paper. Show enough of the reasoning that a reader could follow it without you in the room.

1. A loan, idealized

You take out a loan. You make regular payments on the loan, but at the same time it is accumulating interest. Let's model this by a differential equation for the rate of change of your loan balance $L(t)$.

To write a differential equation we think of the interest as compounding continuously at some rate, and of your payments as flowing out continuously at the steady rate, rather than in monthly chunks.

(a) Write a differential equation for the amount $L(t)$ you owe, in dollars, with t in years (define your own constants for interest and payment rates; and think about which one of these depends *on the current size of the loan* and which does not.)

(b) Without solving anything, read the sign of L' from your equation: for which loan amounts is the debt shrinking, and for which is it growing? What happens at the border in-between?

2. The law of exponents, by uniqueness

Fix a number s, and consider the two functions

$$f(t) = e^{s+t} \qquad \text{and} \qquad g(t) = e^s\, e^t.$$

(a) Verify that both functions solve $y' = y$, and compute $f(0)$ and $g(0)$.

(b) Check that the hypotheses of the existence–uniqueness theorem hold at every point.

(c) What does the theorem now force? State the algebraic identity you have just proved.

(d) Run the same argument again to prove the law of logarithms: for a fixed $s > 0$, show that $\ln(st)$ and $\ln s + \ln t$ solve the same initial-value problem, and conclude that $\ln(st) = \ln s + \ln t$ for every $t > 0$.

3. Drawing a slope field

Consider the equation

$$y' = y^2 - t.$$

(a) Find every point where the equation prescribes slope zero, and describe this set of points as a curve in the (t, y)-plane.

(b) On the region $0 \le t \le 4$, $-2 \le y \le 2$, compute the prescribed slope at each integer grid point and sketch the slope field. Your curve from the previous part organizes the picture: which side of it has rising ticks, and which falling?

(c) Into your field, sketch the solutions starting from $y(0) = 2$ and from $y(0) = 0$. Describe in a sentence or two how their fates differ.

4. Solutions and impostors

In each part, decide whether the proposed function solves the differential equation. Explain your reasoning.

(a) The equation $y' = y + e^t$, with the two candidates $y = te^t$ and $y = 2e^t$.

(b) The system $f' = f + g, g' = g - f$, with the two candidate pairs $f = e^t \cos t, g = -e^t \sin t$ and $f = e^t, g = 0$.

5. Autonomous or not

Consider the three rules

$$\text{(i) } y' = y^2 - 4, \qquad \text{(ii) } y' = y + t, \qquad \text{(iii) } f' = f + g,\ g' = g - f.$$

(a) Which of these are autonomous? Answer with a sentence each.

(b) For rule (ii): show that the state $y = 0$ receives different instructions at two different times. Explain why this makes a phase line for (ii) impossible.

6. A phase line

Consider the autonomous equation

$$y' = y^2 - 4.$$

(a) Find every value of y where the equation prescribes no motion, and draw the phase line, with arrows determined by the sign of $y^2 - 4$.

(b) Using only your phase line, describe the entire future of the solution starting at $y(0) = 0$, and of the solution starting at $y(0) = 3$.

7. Choosing the right guess

(a) Consider $t^2y'' = 2y$. First try the guess $y = e^{rt}$ and explain why *no* value of r can ever work. Then try a power, $y = t^n$: find every value of n that works, and verify the resulting solutions by substitution.

(b) Consider $y' + y = e^{2t}$. Here an exponential is clearly the right shape — but with a knob: try $y = Ae^{2t}$, and find the value of A that makes it a solution.

8. Separating variables

Solve each initial-value problem. State its *domain*: the largest interval containing the initial time on which it solves the differential equation (be careful about division by zero, logarithms of negative numbers, etc.).

(a) $y' = -\dfrac{t}{y}, \quad y(0) = 2.$

(b) $y' = ty + t, \quad y(0) = 0.$

(c) $y' = e^{t+y}, \quad y(0) = 0.$