

Study I

Due Friday, September 18

Write a study guide for yourself in preparation for Exam 1. The exam covers Chapters 1–3 and basic modeling, including the population models at the beginning of Chapter 4.

Use the topics below to decide what belongs in your guide. Look up the important definitions and theorem statements, explain them in your own words, and work examples by hand. For a theorem, include its hypotheses: when can you use it, and what does it let you conclude? For a calculation, explain why the method works and check your answer. Use the notes and Problem Sets 1 and 2 to find examples. Make a note of anything you cannot yet explain or do without looking at a solution; those are the topics to spend more time on.

You should be able to do the calculus calculations from these chapters. You should understand the main ideas of the proofs listed below from Chapter 3. To check your understanding, try explaining why the results are true in your own words. It is okay if your explanation is not perfectly rigorous.

Differential Equations and Their Solutions

- Know what a differential equation is, what its order means, and what it means for a function to be a solution on an interval.
- Know what an initial-value problem is. Be able to check a proposed solution by differentiation and substitution, including checking the initial data. Practice this for single equations and systems.
- Be able to draw a slope field and sketch solution curves through given points. Can you tell from the equation where a solution rises, falls, or has a horizontal tangent?
- Be able to read the graphs of two functions solving a system together: how does the value of one function affect the derivative of the other?

Finding Solutions

- Be able to find solutions by guessing and checking. Practice exponential, power, and trigonometric guesses, including determining constants that make a proposed function satisfy the equation.

- Be able to solve $y' = f(t)$ with an initial condition. Know how to write the answer as a definite integral, even when you cannot find an elementary antiderivative. How does the Fundamental Theorem of Calculus check it?
- Know what a separable equation is and be able to solve one. Explain how separation uses the chain rule. What solutions might you lose by dividing, and how do you check for them?
- Be comfortable with explicit and implicit answers. Be able to recover an explicit solution when possible, choose the correct branch using the initial condition, and check an implicit answer by differentiation.
- Know what a first-order linear equation is. Be able to solve one by undoing the product rule, multiplying the equation by a suitable function when needed. Practice examples with variable coefficients too.
- Be able to make a substitution that simplifies an equation, as in Problem Set 2. Remember to translate the initial condition and return to the original unknown at the end.
- Practice the calculus needed for these methods: substitution, integration by parts, partial fractions, and working with logarithms and exponentials. Be able to differentiate your answers to check them.
- For every initial-value problem you solve, find the largest interval containing the initial time on which your answer solves the equation. Include examples where the solution becomes unbounded in finite time and examples where the equation itself is undefined at an endpoint.

Existence and Uniqueness

- Understand the difference between proving that a solution exists and proving that it is unique. Why does checking a proposed solution only establish the first of these?
- Know the statement of the existence-and-uniqueness theorem for first-order linear equations. Given an equation and initial data, be able to identify the interval on which this theorem guarantees a solution.
- Understand the linear proof we worked through in class. How do we construct a solution using the product rule? How do we check it? What equation does the difference of two solutions satisfy, and how does this prove uniqueness?
- Know the statement of the existence-and-uniqueness theorem for separable equations, including what it says when the initial value is an equilibrium.
- To really understand the separable theorem, you should be able to explain why the relation obtained by integrating determines a unique solution near the initial time.

- Know the statement of the general local existence-and-uniqueness theorem. Be able to calculate the partial derivative with respect to y , check the hypotheses near an initial point, and state the conclusion.
- Know what is meant by local existence. Why can a smooth equation defined everywhere have a solution that ends in finite time? Review an example and explain why this does not contradict the theorem.
- Understand what happens when a theorem's hypotheses fail: does that mean its conclusion is false? Know the chapter's examples where no solution exists and where more than one solution exists, and be able to verify what goes wrong in each.

Understanding Solutions Without Finding Formulas

- Know the uniqueness consequence that two distinct solutions cannot touch. Understand why we can treat a possible meeting point as an initial condition, and why the argument works backward as well as forward in time. What does this tell us about the order of two solutions?
- Know what an autonomous equation and an equilibrium solution are. Be able to find the equilibria and determine the directions of motion between them.
- Draw the equilibrium solutions and sketch curves trapped between them. Show which way the solutions move and which equilibrium they approach. Understand how uniqueness keeps them from touching or crossing the equilibrium solutions.
- Review the chapter's example $y' = t - y$. Check that $y = t - 1$ is a solution and explain why other solutions cannot cross it. Why doesn't uniqueness prevent solutions from crossing the line $y = t$?
- Know the nonoscillation result for smooth scalar autonomous equations and understand its proof. Why would a turning point force a solution to be constant? Why does this argument apply to autonomous equations but not to differential equations more generally?

Modeling Examples

Understand these examples and be able to do problems like them. Be able to turn a description into a differential equation, explain its terms, and use the equation to answer questions about the model.

- The loan problem on Problem Set 1: interest, payments, and when the amount owed grows or shrinks.

- The population section at the beginning of Chapter 4: exponential growth, logistic growth, and harvesting. Understand how the assumptions lead to the equations and how to investigate their predictions using solutions, equilibria, and signs.