Study III

The exam covers mainly Chapters 7–10. **You are only responsible for the topics on this study guide.**

Use the topics below to prepare your guide. Explain the important ideas in your own words and work examples by hand. For a calculation, explain why the method works and check your answer. Use the notes and Problem Sets 5–7 to find examples.

REVIEW MATERIAL

- Be able to check a proposed solution by differentiation and substitution, including checking its initial conditions.
- Know why solutions of a homogeneous linear equation form a vector space. Be able to find its dimension from the number of initial conditions.
- Given a basis of solutions, be able to choose coefficients to satisfy initial conditions. Know why this gives every solution.
- Be able to convert a higher-order differential equation into a first-order system and write a linear system in matrix form.

The Matrix Exponential

- Know the series definition of e^{tA}. Be able to derive its coefficients from $\Phi' = A\Phi$, $\Phi(0) = I$.
- Be able to solve $X' = AX$, $X(0) = X_0$, using $X(t) = e^{tA}X_0$.
- Be able to compute e^{tA} by finding a pattern in the powers of A. Practice diagonal matrices, matrices whose powers eventually vanish, and matrices whose powers produce sine, cosine, or hyperbolic functions.
- Be able to use the series to explain why A commutes with e^{tA}.
- Know the commuting-sum law: if $BC = CB$, then $e^{t(B+C)} = e^{tB}e^{tC}$. Understand its proof using uniqueness and where the commuting hypothesis is needed.
- Know the laws $e^{(s+t)A} = e^{sA}e^{tA}$ and $(e^{tA})^{-1} = e^{-tA}$. Explain how they describe advancing or reversing the flow.

- Be able to compute an exponential by splitting a matrix into commuting pieces, including splitting off a multiple of the identity.

Linear Flows in the Plane

- Be able to find eigenvalues and eigenvectors. Know why $Av = \lambda v$ gives the solution $X(t) = e^{\lambda t}v$.
- Be able to change coordinates using $X = PY$, giving $Y' = P^{-1}APY$. Transform the initial condition and return to the original coordinates after solving.
- Know why $A = PCP^{-1}$ gives $e^{tA} = Pe^{tC}P^{-1}$.
- For a matrix with two independent real eigenvectors, be able to diagonalize it and solve an initial-value problem.
- For a matrix with only one independent real eigenvector, be able to compute its exponential using shear and scaling. Explain where the factor $te^{\lambda t}$ comes from.
- For a matrix with no real eigenvectors, be able to compute its exponential using rotation and scaling. Find the scaling rate, angular frequency, and period of oscillation.
- Be able to recognize and sketch sinks, sources, saddles, closed orbits, spirals going in, and spirals going out. Show the direction of motion.
- Be able to classify a planar system using its eigenvalues. Trace, determinant, and the discriminant can provide a shortcut.
- Review the example on Problem Set 6 where a solution first moves away from the origin and later approaches it.
- Be able to convert $x'' + bx' + kx = 0$ into a first-order system and solve it by matrix exponentiation.

Linear Flows in Higher Dimensions

- When the eigenvectors form a basis, be able to write the initial state as a combination of eigenvectors and evolve each term by its exponential.
- Be able to factor a characteristic polynomial into real linear and quadratic factors. A linear factor $r - \lambda$ gives the rate λ. Complete the square in a quadratic factor to find the scaling rate a and angular frequency ω in $(r - a)^2 + \omega^2$.
- Be able to carry out the rotation-plane construction from Problem Set 7. Find the vectors v, w and check the equations $Av = av - \omega w, \qquad Aw = \omega v + aw$.

  Use these equations to write the 2×2 matrix describing motion in that plane.

- Be able to solve a three-dimensional system with an eigenvector line and a rotation plane. Express the initial state in the basis consisting of the eigenvector and the two plane vectors, solve in those coordinates, and change back.
- Be able to exponentiate a block-diagonal matrix by exponentiating each block. Explain why this works using powers of the matrix.
- Be able to compute the exponential of a matrix with a constant diagonal and ones immediately above it. Write it as $\lambda I + N$, compute powers of N until they vanish, and use the terminating exponential series.
- Be able to compute the exponential of the coupled rotation blocks on Problem Set 7 by splitting the matrix into commuting pieces.
- Be able to determine which initial states give solutions that approach the origin and which give growing solutions. Find the rates of growth or decay and the periods of any oscillations.

Modes

- Be able to write the equations for masses connected by springs, including connections to fixed supports.
- Be able to convert $q'' = -Kq$ into a first-order system using displacement and velocity as the state.
- Be able to derive $C_K(t)$ and $S_K(t)$ from the matrix exponential by computing powers of the first-order matrix and collecting the even and odd terms.
- Be able to solve an initial-value problem using $q(t) = C_K(t)q_0 + S_K(t)p_0$ and check both initial conditions.
- Know why $Kv = \lambda v$ reduces a solution $q(t) = a(t)v$ to $a'' = -\lambda a$. For $\lambda > 0$, find the frequency and period, and interpret the entries of v as relative displacements.
- Be able to use the series to show that, when $Kv = \omega^2 v$, $C_K(t)v = \cos(\omega t)v$ and $S_K(t)v = \sin(\omega t)v/\omega$.
- Know why a mode shape v gives the invariant plane spanned by $(v, 0)$ and $(0, v)$ in the first-order system.
- Be able to write an arbitrary state as a sum of modes. Know why a basis of n mode shapes with positive eigenvalues gives $2n$ independent solutions.