

Study II

Due Monday, October 12

Write a study guide for yourself in preparation for Exam 2. The exam covers Chapters 5–7 and the first part of Chapter 8. The instructions are the same as for Study I. Use the topics below to decide what belongs in your guide, and use the notes and Problem Sets 3–5 to find examples. The exam focuses on these topics, but you are still expected to know the material from the last exam.

Linear Algebra

- Know what a vector space and a subspace are, and how to check whether a set of vectors or functions is or is not a subspace.
- Know what linear independence, a basis, and dimension mean. Be able to check whether a list of vectors or functions is linearly independent.
- Know what a linear map is, and what its kernel and image are. Be able to state the rank–nullity theorem.
- Be able to write the matrix of a linear map in chosen bases: its columns are the coordinates of the images of the basis vectors. Know why composing maps corresponds to multiplying matrices, and why an isomorphism has an invertible matrix.
- Know what an affine space is, and why the solutions of a linear equation $T(v) = w$ form one: a particular solution plus the kernel.

Equations and Solution Spaces

- Know what the solution set of a differential equation is, and what its ambient space is. We know the solution sets of $y' = y$ and of the spring $x'' = -x$ exactly. Be able to explain why the first is a line and the second is a plane in a space of functions.
- The logistic equation $y' = y(1 - y)$ also has one solution for each initial value $y(0) \in \mathbb{R}$, so its solutions form a one-parameter family. Why isn't that family a line? Check that $y = 1$ is a solution but $y = 2$ is not.
- Understand why spaces of functions such as $C^k(I)$ are vector spaces, and why they are infinite-dimensional.

- Be able to write any equation in first-order form: choose the state, write the system, and say what an initial condition prescribes. Practice higher-order equations like $x''' = x''$ and systems that mix different orders.
- Know what the initial-condition space is, and be able to find it for a given equation.
- Know the existence and uniqueness theorem for systems.
- Know the evaluation map, and why uniqueness makes it one-to-one. Why can it fail to be onto? Know an example.
- Know what the solution map is, and on which initial conditions it is defined.
- Know how to use the existence and uniqueness theorem to prove that you have found *all* solutions of a differential equation, once you have found one for every possible initial condition.

Linear Evolution

- Know what a linear operator is, and be able to check whether a given operator is linear.
- Know what a linear differential equation is, written as $L[y] = g$, and what homogeneous and forced mean. Be able to decide whether a given equation is linear.
- Be able to write a linear first-order system in matrix form, $X' = A(t)X + g(t)$, and read off $A(t)$ and $g(t)$.
- Know why the solutions of a homogeneous linear equation form a vector space, and why the solutions of a forced linear equation form an affine space $u_p + \ker L$.
- Know the existence and uniqueness theorem for linear systems. What does it promise that the general theorem does not?
- Know why evaluation is a linear isomorphism for a homogeneous linear equation, so that $\dim \mathcal{S} = \dim \mathcal{J}$. Be able to find the dimension of the solution space without solving the equation.
- Know how to find every solution of a homogeneous linear equation by finding $\dim \mathcal{J}$ independent solutions, and of a forced one by adding one particular solution.
- Be able to check whether solutions form a basis by checking whether their initial conditions do.
- Given a basis of solutions, be able to write the matrix M of the evaluation map and use M^{-1} to solve initial value problems.
- Know what the standard solutions are, and why their evaluation matrix is the identity.

- Know what the flow matrix is, how it is built from the standard solutions, and why $X(t) = \Phi(t)X_0$.
- Be able to find the flow matrix of the spring, or of other simple systems where you can find the standard solutions by calculus or by guessing and checking.

The Matrix Exponential

- Know why the flow matrix of a linear system satisfies the matrix equation $\Phi' = A(t)\Phi$, $\Phi(t_0) = I$.
- Know how to find the power series of the exponential from the differential equation $y' = ay$, $y(0) = 1$, by matching coefficients. Why does this series define e^x for every real x , and not just for rational x ?
- Know the definition of the matrix exponential e^{tA} , and how the same matching of coefficients produces it.
- Know why e^{tA} is the flow matrix of $X' = AX$, so that the solution from $X(t_0) = X_0$ is $e^{(t-t_0)A}X_0$.
- Know how uniqueness proves the law of exponents $e^{(t+s)A} = e^{tA}e^{sA}$, and why it gives $(e^{tA})^{-1} = e^{-tA}$.
- Be able to compute e^{tA} when the powers of A follow a pattern: multiples of I , diagonal matrices, matrices whose powers eventually vanish, and matrices like J whose powers cycle, which produce cosine and sine.
- Know that $e^{t(B+C)}$ need not equal $e^{tB}e^{tC}$ when $BC \neq CB$.

The Geometry of Linear Flows

- Know what the eigenvalues and eigenvectors of a matrix are, and be able to find them for a 2×2 matrix using the characteristic polynomial.
- Know why an eigenvector gives a straight-line solution: if $Av = \lambda v$, then $X(t) = e^{\lambda t}v$ solves $X' = AX$.
- Know how to solve $X' = AX$ by decoupling: when A has a basis of eigenvectors, write the state in eigenvector coordinates, where each coordinate evolves on its own.
- Be able to use eigenvectors to exponentiate a matrix: if $A = PDP^{-1}$ with D diagonal, then $e^{tA} = Pe^{tD}P^{-1}$.